Summer Challenge (due August 31)

Fix a circle Γ. Let $T=\triangle ABC$ be a triangle inscribed in the circle Γ which is not a right triangle and let H be the orthocenter of T. The line HA intersects the circle Γ at A and at a second point A_1 (which can be A if the line is tangent to the circle). Likewise, the line HB intersects Γ at a second point B_1, and the line HC intersects Γ at a second point C_1. The triangle $\triangle A_1B_1C_1$ is again inscribed in Γ. We denote this triangle by $\Phi(T)$. Warning: $\Phi(T)$ can be a right triangle.

a) Show that triangles T and $\Phi(T)$ are congruent if and only if either T is equilateral or the angles of T are $\pi/7$, $2\pi/7$, $4\pi/7$.

b) For every integer $k>0$ find the number t_k of non-congruent triangles T inscribed in Γ such that $\Phi^k(T)$ and T are congruent. Here Φ^k denotes the composition $\Phi\circ \Phi \circ \ldots \circ\Phi$ of Φ with itself k times. Thus, according to a), we have $t_1=2$.

c) Is it true that if $\Phi^k(T)$ and T are congruent then $\Phi^m(T)=T$ for some m?

d) Find and prove your own results about Φ.